Exponential utility maximization under partial information and sufficiency of information

Marina Santacroce

Politecnico di Torino

Joint work with M. Mania

WORKSHOP “FINANCE and INSURANCE”
March 16-20, Jena 2009
Outline

Expected utility maximization under partial information: semimartingale setting
  Semimartingale model
  Expected utility and partial information
  Equivalent problem and solution

Value process and BSDE
  Assumptions
  Equivalent problem
  Value process and BSDE
  \( \mathcal{F}^S \subseteq \mathcal{G} \)

An example with explicit solution.
  Diffusion model with correlation

Sufficiency of filtrations
  Sufficiency of filtrations
The model

- Let $S = (S_t, t \in [0, T])$ be a continuous semimartingale which represents the price process of the traded asset.
- $(\Omega, \mathcal{A}, \mathcal{A} = (\mathcal{A}_t, t \in [0, T]), P)$, where $\mathcal{A} = \mathcal{A}_T$ and $T < \infty$ is a fixed time horizon.
- Assume the interest rate equal to zero.

The price process $S$ admits the decomposition

$$S_t = S_0 + N_t + \int_0^t \lambda_u d\langle N \rangle_u, \quad \langle \lambda \cdot N \rangle_T < \infty \quad \text{a.s.,}$$

where $N$ is a continuous $\mathcal{A}$-local martingale and $\lambda$ is a $\mathcal{A}$-predictable process (Structure condition).
Expected utility maximization and partial information

Denote by \( \mathcal{G} = (\mathcal{G}_t, t \in [0, T]) \) a filtration smaller than \( \mathcal{A} \)

\[ \mathcal{G}_t \subseteq \mathcal{A}_t, \quad \text{for every} \quad t \in [0, T]. \]

\( \mathcal{G} \) represents the information available to the hedger.

We consider the exponential utility maximization problem with random payoff \( H \) at time \( T \) when \( \mathcal{G} \) is the available information,

\[
\text{to maximize} \quad E [-e^{-\alpha (x + \int_0^T \pi_u dS_u - H)}] \quad \text{over all} \quad \pi \in \Pi(\mathcal{G}).
\]

- \( x \) represents the initial endowment (w.l.g. we take \( x = 0 \))
- \( \Pi(\mathcal{G}) \) is a certain class of self-financing strategies (\( \mathcal{G} \)-predictable and \( S \)-integrable processes).
- \( (\int_0^t \pi_u dS_u, t \in [0, T]) \) represents the wealth process related to the self-financing strategy \( \pi \).

This problem is equivalent to

\[
\text{to minimize} \quad E [e^{-\alpha (\int_0^T \pi_u dS_u - H)}] \quad \text{over all} \quad \pi \in \Pi(\mathcal{G}).
\]
In most papers, under various setups, (see, e.g., Lakner (1998), Pham and Quenez (2001), Zohar (2001)) expected utility maximization problems have been considered for market models where only stock prices are observed, while the drift can not be directly observed.

⇒ under the hypothesis $\mathcal{F}^S \subseteq \mathcal{G}$.

We consider the case when $\mathcal{G}$ does not necessarily contain all information on the prices of the traded asset i.e.

$S$ is not a $\mathcal{G}$-semimartingale in general!

⇒ In this case, we solve the problem in 2 steps:

- Step 1: Prove that the expected exponential maximization problem is equivalent to another maximization problem of the filtered terminal net wealth (reduced problem)
- Step 2: Apply the dynamic programming method to the reduced problem.

(In Mania et al. (2008) a similar approach is used in the context of mean variance hedging).
Filtration $\mathcal{F}$ and decomposition of $S$ w.r.t. $\mathcal{F}$

- Let $\mathcal{F} = (\mathcal{F}_t, t \in [0, T])$ be the augmented filtration generated by $\mathcal{F}^S$ and $\mathcal{G}$.

- $S$ is a $\mathcal{F}$-semimartingale:

  $$S_t = S_0 + \int_0^t \hat{\lambda}_u(\mathcal{F}) d\langle M \rangle_u + M_t,$$

  (Decomposition of $S$ with respect to $\mathcal{F}$)

  $$M_t = N_t + \int_0^t [\lambda_u - \hat{\lambda}_u(\mathcal{F})] d\langle N \rangle_u$$

  is $\mathcal{F}$-local martingale

  where we denote by $\hat{\lambda}(\mathcal{F})$ the $\mathcal{F}$-predictable projection of $\lambda$.

- Note that $\langle M \rangle = \langle N \rangle$ are $\mathcal{F}^S$-predictable.
Assumptions

In the sequel we will make the following assumptions:

A) $\langle M \rangle$ is $\mathcal{G}$-predictable and $d\langle M \rangle_t dP$ a.e. $\hat{\lambda}^\mathcal{F} = \hat{\lambda}^\mathcal{G}$, hence for each $t$

$$E(\lambda_t | \mathcal{F}^S_t \vee \mathcal{G}_t) = E(\lambda_t | \mathcal{G}_t), \quad P - \text{a.s.}$$

B) any $\mathcal{G}$-martingale is a $\mathcal{F}$-local martingale,

C) the filtration $\mathcal{G}$ is continuous,

D) for any $\mathcal{G}$-local martingale $m(g) \langle M, m(g) \rangle$ is $\mathcal{G}$-predictable,

E) $H$ is an $\mathcal{A}_T$-measurable bounded random variable, such that $P$- a.s.

$$E[e^{\alpha H} | \mathcal{F}_T] = E[e^{\alpha H} | \mathcal{G}_T],$$
Remarks

Note that

- If $\mathcal{F}^S \subseteq \mathcal{G}$, then $\langle M \rangle$ is $\mathcal{G}$-predictable. Besides, in this case $\mathcal{G} = \mathcal{F}$ and conditions $A), B), D)$ and the equality in $E)$ are automatically satisfied.

- Condition $B)$ is satisfied if and only if the $\sigma$-algebras $\mathcal{F}_t^S \vee \mathcal{G}_t$ and $\mathcal{G}_T$ are conditionally independent given $\mathcal{G}_t$ for all $t \in [0, T]$ (Jacod 1978).

- The continuity of filtration $\mathcal{G}$ is weaker than the assumption that the filtration $\mathcal{F}$ is continuous.
  
  Filtration $\mathcal{F}$ continuous + condition $B) \Rightarrow$ Filtration $\mathcal{G}$ is continuous

  The converse is not true in general.

- The filtrations $\mathcal{F}^S$ and $\mathcal{F}$ can be discontinuous.
$\mathcal{G}$-projection processes

Let $\hat{S}_t = E(S_t|\mathcal{G}_t)$ be the $\mathcal{G}$-optional projection of $S_t$. Note that since $\hat{\lambda}^\mathcal{F} = \hat{\lambda}^\mathcal{G} = \hat{\lambda}$, we have:

$$\hat{S}_t = E(S_t|\mathcal{G}_t) = S_0 + \int_0^t \hat{\lambda}_u d\langle M \rangle_u + \hat{M}_t$$

where $\hat{M}_t$ is the $\mathcal{G}$-local martingale $E(M_t|\mathcal{G}_t)$.

- Under conditions A-D,

  $$\hat{M}_t = \int_0^t \frac{d\langle M, m(g) \rangle_u}{d\langle m(g) \rangle_u} dm_u(g) + L_t(g),$$

  where $m(g)$ is any $\mathcal{G}$-local martingale and $L(g)$ is a $\mathcal{G}$-local martingale orthogonal to $m(g)$.

- Hence, $\langle M, m(g) \rangle_t = \langle \hat{M}, m(g) \rangle_t \Rightarrow \langle M, \hat{M} \rangle_t = \langle \hat{M} \rangle_t$.

Using this decomposition and the Doléans equation, it is possible to show

**Lemma** Under conditions A)-D),

$$\mathcal{E}_t(M) = E(\mathcal{E}_t(M)|\mathcal{G}_t) = \mathcal{E}_t(\hat{M}).$$
Equivalent problem

We recall that our aim is

to minimize \( E[e^{-\alpha(\int_0^T \pi_u dS_u - H})] \) over all \( \pi \in \Pi(\mathcal{G}). \) (1)

where the class of strategies is defined as

\[
\Pi(\mathcal{G}) = \{ \pi : \mathcal{G} - \text{predictable}, \pi \cdot M \in BMO(\mathcal{F}) \}
\]

**PROPOSITION** Let conditions A)-E) be satisfied. Then the optimization problem (1) is equivalent to

\[
\text{to minimize } E[e^{-\alpha(\int_0^T \pi_u d\hat{S}_u - \tilde{H})} + \frac{\alpha^2}{2} \int_0^T \pi_u^2 (1 - \kappa_u^2) d\langle M\rangle_u], \text{ over all } \pi \in \Pi(\mathcal{G}) \quad (2)
\]

\[
\tilde{H} = \frac{1}{\alpha} \ln E[e^{\alpha H}|\mathcal{G}_T], \quad \kappa_t^2 = \frac{d\langle \hat{M}\rangle_t}{d\langle M\rangle_t}.
\]

Moreover, for any \( \pi \in \Pi(\mathcal{G}) \)

\[
E[e^{-\alpha(\int_t^T \pi_u dS_u - H})|\mathcal{G}_t] = E[e^{-\alpha(\int_t^T \pi_u d\hat{S}_u - \tilde{H})} + \frac{\alpha^2}{2} \int_t^T \pi_u^2 (1 - \kappa_u^2) d\langle M\rangle_u |\mathcal{G}_t].
\]
Sketch of the proof

Let $t = 0$. Taking the conditional expectation with respect to $\mathcal{F}_T$ and using condition $E[e^{\alpha H}|\mathcal{F}_T] = E[e^{\alpha H}|\mathcal{G}_T]$, we have that

$$E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] = E\left(e^{-\alpha \int_0^T \pi_u dS_u} E[e^{\alpha H}|\mathcal{F}_T]\right) = E\left(e^{-\alpha \int_0^T \pi_u dS_u - \tilde{H}}\right).$$

Resorting to

- the $\mathcal{F}$-decomposition of $S$,
- the previous lemma,
- the decomposition of $\hat{S}$,

we have that (3) is equal to

$$E\left[\mathcal{E}_T(-\alpha \pi \cdot M) e^{\frac{\alpha^2}{2} \int_0^T \pi_u^2 d\langle M \rangle_u - \alpha \int_0^T \pi_u \hat{\lambda}_u d\langle M \rangle_u + \alpha \tilde{H}}\right]$$

$$= E\left[E\left(\mathcal{E}_T(-\alpha \pi \cdot M)|\mathcal{G}_T\right) e^{\frac{\alpha^2}{2} \int_0^T \pi_u^2 d\langle M \rangle_u - \alpha \int_0^T \pi_u \hat{\lambda}_u d\langle M \rangle_u + \alpha \tilde{H}}\right]$$

$$= E\left[\mathcal{E}_T(-\alpha \pi \cdot \hat{M}) e^{\frac{\alpha^2}{2} \int_0^T \pi_u^2 d\langle M \rangle_u - \alpha \int_0^T \pi_u \hat{\lambda}_u d\langle M \rangle_u + \alpha \tilde{H}}\right]$$

$$= E\left[e^{-\alpha \int_0^T \pi_u d\hat{M}_u - \frac{\alpha^2}{2} \int_0^T \pi_u^2 d\langle \hat{M} \rangle_u + \frac{\alpha^2}{2} \int_0^T \pi_u^2 d\langle M \rangle_u - \alpha \int_0^T \pi_u \hat{\lambda}_u d\langle M \rangle_u + \alpha \tilde{H}}\right]$$

$$= E\left[e^{-\alpha (\int_0^T \pi_u d\hat{S}_u - \tilde{H}) + \frac{\alpha^2}{2} \int_0^T \pi_u^2 (1-\kappa_u^2) d\langle M \rangle_u}\right].$$
Remarks

- The previous proposition says that the optimization problems (1) and (2) are equivalent.
- It is sufficient to solve problem (2), which is formulated in terms of $\mathcal{G}$-adapted processes.
- We can see (2) as an exponential hedging problem under complete information with a (multiplicative) correction term and we can solve it using methods for complete information.

Let

$$V_t = \operatorname{ess inf}_{\pi \in \Pi(\mathcal{G})} E[e^{-\alpha(\int_t^T \pi_u d\tilde{S}_u - \tilde{H}) + \frac{\alpha^2}{2} \int_t^T \pi_u^2 (1 - \kappa_u^2) d\langle M \rangle_u} | \mathcal{G}_t],$$

be the value process related to the equivalent problem.
From now on let assume
\[ F) \quad \int_0^T \hat{\lambda}_t^2 d\langle M \rangle_t \leq C, \quad P - \text{a.s.}. \]

Note that:

- \( c \leq V_t \leq C \) for two positive constants:
  - \( H \) is bounded \( \Rightarrow V_t \leq C \).
  - condition F) \( \Rightarrow V_t \geq c \) for a positive c (by duality issues).

Indeed, if \( V_t(\mathcal{F}) = \text{ess inf}_{\pi \in \Pi(\mathcal{F})} E[e^{-\alpha(\int_T^t \pi_u dS_u - H)} | \mathcal{F}_t] \), then it is well known that under condition F), \( V_t(\mathcal{F}) \geq c \), (see e.g. Delbaen et al. (2002)). Therefore

\[
V_t = \text{ess inf}_{\pi \in \Pi(\mathcal{G})} E \left[ E \left( e^{-\alpha(\int_T^t \pi_u dS_u - H)} | \mathcal{F}_t \right) | \mathcal{G}_t \right] \geq E(V_t(\mathcal{F}) | \mathcal{G}_t) \geq c.
\]

- If \( H \) is \( \mathcal{G}_T \)-measurable, then \( \tilde{H} = H \) and problem (1) is equivalent to minimize

\[
E[e^{-\alpha(\int_0^T \pi_u dS_u - H)} + \frac{\alpha^2}{2} \int_0^T \pi_u^2(1 - \kappa_u^2)d\langle M \rangle_u].
\]
Main result

**Theorem** Under assumptions A)-F). Then, the value process $V$ related to the equivalent problem (2) is the unique bounded strictly positive solution of the following BSDE

$$
Y_t = Y_0 + \frac{1}{2} \int_0^t \frac{(\psi_u \kappa_u^2 + \lambda_u Y_u)^2}{Y_u} d\langle M \rangle_u + \int_0^t \psi_u d\hat{M}_u + L_t
$$

$$Y_T = E[e^{\alpha H} | \mathcal{G}_T]$$

Moreover the optimal strategy exists in the class $\Pi(\mathcal{G})$ and is equal to

$$
\pi_t^* = \frac{1}{\alpha} (\lambda_t + \psi_t \kappa_t^2). \tag{5}
$$

- The existence of a solution to the BSDE (4) is derived by using results in Tevzadze (2008) (see also Morlais (2008)).
- Unicity is proved by directly showing that the unique bounded solution of the BSDE is the value process of the problem.
What happens if $\mathcal{F}_t^S \subseteq \mathcal{G}_t$?

- $\mathcal{G}_t = \mathcal{F}_t \equiv \mathcal{F}_t^S \lor \mathcal{G}_t$

The “$\mathcal{F}$ decomposition” of $S$ is the “$\mathcal{G}$ decomposition of $S$”:

$$S_t = S_0 + \int_0^t \lambda_u d\langle M \rangle_u + M_t, \quad M \in \mathcal{M}_{\text{loc}}(\mathcal{G})$$

Note

- $\tilde{M}_t = M_t$ and $\kappa_t^2 = 1$ for all $t$
- A), B), D) and $E[e^{\alpha H} | \mathcal{F}_T] = E[e^{\alpha H} | \mathcal{G}_T]$ are satisfied

$\implies$ If $\mathcal{G}$ is continuous, the initial problem (1) is equivalent to

$$\text{to minimize} \quad E(e^{-\alpha \int_0^T \pi_u dS_u - \tilde{H}}) \quad \text{over all} \quad \pi \in \Pi(\mathcal{G}).$$
**Corollary**  Let $\mathcal{F}^S \subseteq \mathcal{G} \subseteq \mathcal{A}$. Assume $\mathcal{G}$ continuous, $H$ to be a bounded $\mathcal{A}_T$-measurable random variable, $\int_0^T \hat{\lambda}_t^2 d\langle M \rangle_t \leq C, \ P - \text{a.s.}$. Then, the value process $V$ is the unique bounded positive solution of the BSDE

$$
Y_t = Y_0 + \frac{1}{2} \int_0^t \left( \psi_u + \hat{\lambda}_u Y_u \right)^2 \frac{d\langle M \rangle_u}{Y_u} + \int_0^t \psi_u dM_u + L_t, \quad Y_T = E(e^{\alpha H} | \mathcal{G}_T). \tag{6}
$$

Moreover, the optimal strategy is equal to

$$
\pi^*_t = \frac{1}{\alpha} \left( \hat{\lambda}_t + \frac{\psi_t}{Y_t} \right).
$$

**Remark:** In the case of full information $\mathcal{G}_t = \mathcal{A}_t$, we also have

$$
\hat{M}_t = M_t = N_t, \quad \hat{\lambda}_t = \lambda_t, \quad Y_T = e^{\alpha H}
$$

and Equation (6) takes on the form

$$
Y_t = Y_0 + \frac{1}{2} \int_0^t \left( \psi_u + \lambda_u Y_u \right)^2 \frac{d\langle N \rangle_u}{Y_u} + \int_0^t \psi_u dN_u + L_t, \quad Y_T = e^{\alpha H}.
$$
As an example we deal with a diffusion market model consisting of two correlated risky assets one of which has no liquid market.

- First we consider an agent who is finding the best strategy to hedge a contingent claim $H$, trading with the liquid asset but using just the information on the non tradable one (partial information).
- We solve the problem for full information when $H$ is function only of the nontraded asset.
- Then, we compare the results obtained in the cases of partial/full information.
Basis risk model

The price of the two risky assets follow the dynamics

\[
\begin{align*}
\tilde{d}S_t &= \tilde{S}_t(\mu(t, \eta)dt + \sigma(t, \eta)dW^1_t), \\
d\eta_t &= b(t, \eta)dt + a(t, \eta)dW_t,
\end{align*}
\]

subjected to initial conditions.

- \(W^1\) and \(W\) are standard Brownian motions with constant correlation \(\rho\), i.e. \(E dW^1_t dW_t = \rho dt\), \(\rho \in (-1, 1)\).

\[
W_t = \rho W^1_t + \sqrt{1 - \rho^2} W^0_t,
\]

where \(W^0\) and \(W^1\) are independent Brownian motions.
- \(\eta\) represents the price of a nontraded asset (e.g. an index);
- Besides, we denote

\[
dS_t = \mu(t, \eta)dt + \sigma(t, \eta)dW^1_t
\]

\[
dS_t = \frac{\tilde{d}S_t}{\tilde{S}_t} \Rightarrow S \text{ represents the process of returns of the tradable asset.}
\]
Assumptions

Assume that the coefficients $\mu$, $\sigma$, $a$ and $b$ are non anticipative functionals such that:

1) $\int_0^T \frac{\mu^2(t,\eta)}{\sigma^2(t,\eta)} dt$ is bounded,

2) $\sigma^2 > 0$, $a^2 > 0$

3) Eq. (7) admits a unique strong solution $(\eta)$,

4) $H$ is bounded and $\mathcal{F}_T^\eta \vee \sigma(\xi)$-measurable, where $\xi$ is a random variable independent of $S$ and $\eta$.

Under conditions 2), 3) we have $\mathcal{F}^{S,\eta} = \mathcal{F}^{W^1,W}$ and $\mathcal{F}^\eta = \mathcal{F}^W$. 
Partial information

Problem: An agent is hedging a contingent claim $H$ trading with the liquid asset $S$ using only observations coming from $\eta$.

to minimize $E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}]$ over all $\pi \in \Pi(F^\eta)$, where $\pi$ represents the dollar amount the agent invests in the stock which depends only on $\eta$.

$\mathcal{F}_t = \mathcal{F}_t^{S,\eta} \subseteq \mathcal{A}_t$ and $\mathcal{G}_t = \mathcal{F}_t^\eta$.

Under conditions 1)–4) (which imply that our assumptions A)–F) are satisfied),

we give an explicit expression of the optimal amount of money that should be invested in the liquid asset.
In this model,

\[ M_t = \int_0^t \sigma(u, \eta) dW_u^1, \quad \langle M \rangle_t = \int_0^t \sigma^2(u, \eta) du, \quad \lambda_t = \frac{\mu(t, \eta)}{\sigma^2(t, \eta)}. \]

\[ \hat{M}_t = \rho \int_0^t \sigma(u, \eta) dW_u \Rightarrow \langle \hat{M} \rangle_t = \rho^2 \langle M \rangle_t \text{ and } \kappa_t^2 = \rho^2. \]

Conditions 1)–4) imply that assumptions A)–F) hold, indeed

- \( \mu, \sigma, b, a \) are \( \mathcal{F}^\eta \)-measurable \( \Rightarrow \) conditions A) and D);
- \( \mathcal{F}^{W^1, W} \) (resp. \( \mathcal{F}^W \)) is the (augmented) filtration generated by \( W^1 \) and \( W \) (resp. \( W \)) \( \Rightarrow \mathcal{G} = \mathcal{F}^\eta \) is continuous (condition C));
- \( W \) is a Brownian motion (also) with respect to \( \mathcal{F}^{W^1, W} \Rightarrow \) condition B);
- since \( \mathcal{F}_T = \mathcal{F}_T^{S, \eta} \), assumption 4) \( \Rightarrow \) condition E);
- condition F) is here represented by 1).
We find that the value process related to the optimization problem is equal to

\[ V_t = \left( E^{\tilde{Q}} \left[ e^{(1-\rho^2)\left(\alpha \tilde{H} - \frac{1}{2} \int_t^T \theta_u^2 du\right)} \middle| F_t \right] \right)^{\frac{1}{1-\rho^2}}. \]  

(8)

The optimal strategy \( \pi^* \) is identified by

\[ \pi^*_t = \frac{1}{\alpha \sigma(t, \eta)} \left( \theta_t + \frac{\rho h_t}{(1-\rho^2)(c + \int_0^t h_u d\tilde{W}_u)} \right), \]  

(9)

- \( \theta \) stands for the market price of risk \( \theta_t = \frac{\mu}{\sigma} \)
- \( h_t \) is the integrand of the integral representation

\[ e^{(1-\rho^2)\left(\alpha \tilde{H} - \frac{1}{2} \int_0^T \theta_u^2 du\right)} = c + \int_0^T h_t d\tilde{W}_t. \]

- \( \tilde{Q} \), defined by \( \frac{d\tilde{Q}}{dP} = \mathcal{E}_T(-\rho \theta \cdot \mathcal{W}) \) is a new measure and \( \tilde{W}_t = W_t + \rho \int_0^t \theta_u du \) is a \( \tilde{Q} \)-Brownian motion.
Problem: An agent is trading with a portfolio of stocks $\tilde{S}$ in order to solve the optimization problem

$$
to minimize \ E \left( e^{-\alpha \int_{0}^{T} \pi_t d\tilde{S}_t - H} \right) \quad over \ all \ \ \ \pi \in \Pi(\mathcal{F}_{\tilde{S},\eta}), \quad (10)
$$

where $\pi_t$ represents the number of stocks held at time $t$ and is adapted to the filtration $\mathcal{F}_{t,\tilde{S},\eta}$ and $H = f(\eta)$ is written on the non traded asset.

- the agent builds his strategy using all market information, and $\mathcal{G}_t = \mathcal{F}_{t,\tilde{S},\eta} = \mathcal{A}_t$;

This problem was earlier studied in, e.g., Musiela and Zariphopoulou (2004), Henderson and Hobson (2005), Davis (2006), for the basis risk model with constant $\mu$ and $\sigma$. Assuming the Markov structure of $b$ and $a$, Musiela and Zariphopoulou gave an explicit expression for the related value function.

- Under suitable conditions on the coefficients of the model [i.e. 1)–3)] and assuming $H = f(\eta)$ and bounded, we show that the value process related to (10) coincides with (8), and that the optimal amount of money depends only on the observation coming from the non traded asset.
Comparison

Therefore, under conditions 1)–3) and assuming $H = f(\eta)$ and bounded

- we recover the result of Musiela and Zariphopoulou in a non Markovian setting
- we find that the \textbf{optimal amount of money} depends only on the observation coming from the non traded asset.

\emph{Indeed, the two optimization problems (partial/full information) are equivalent: the corresponding value processes coincide and the optimal strategies of these problems are related by the equality}

$$\pi^\ast_{\text{partial}} = \tilde{\pi}^\ast_{\text{full}} \tilde{S}_t.$$ 

\implies This means that, if $H = f(\eta)$, the optimal dollar amount invested in the assets is the same in both problems and is based only on the information coming from the non traded asset $\eta$.

\implies In other words, the information coming from $\eta$ is \textbf{sufficient} for the full information problem...
Sufficiency of filtrations

- Let $V_t(\mathcal{A})$ and $V_t(\mathcal{G})$ be the value processes of the problems respectively in the cases of full and partial information, i.e.

$$V_t(\mathcal{A}) = \text{ess inf}_{\pi \in \Pi(\mathcal{A})} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|\mathcal{A}_t],$$

$$V_t(\mathcal{G}) = \text{ess inf}_{\pi \in \Pi(\mathcal{G})} E[e^{-\alpha(\int_t^T \pi_u dS_u - H)}|\mathcal{G}_t].$$

- We express the optimization problem as the set

$$O\Pi = \{ (\Omega, \mathcal{A}, \mathcal{G}, P), S, H \}$$

and we give the following definition of sufficient filtration.

**Definition** The filtration $\mathcal{G}$ is said to be sufficient for the optimization problem $O\Pi$ if $V_0(\mathcal{G}) = V_0(\mathcal{A})$, i.e., if

$$\inf_{\pi \in \Pi(\mathcal{A})} E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}] = \inf_{\pi \in \Pi(\mathcal{G})} E[e^{-\alpha(\int_0^T \pi_u dS_u - H)}]. \quad (11)$$

⇒ Our aim is to give sufficient conditions for (11).
Assumptions and $G$ projections

We recall the $\mathcal{A}$-decomposition of $S$

$$S_t = S_0 + N_t + \int_0^t \lambda_u d\langle N \rangle_u.$$ 

We make the following assumptions, which are similar to the previous ones, but are written with respect to $\mathcal{A}$ instead of $\mathcal{F}$ (so they are stronger):

- $A')$ $\langle N \rangle$ is $G$-predictable,
- $B')$ any $G$-martingale is an $\mathcal{A}$-local martingale,
- $C')$ the filtration $G$ is continuous,
- $D')$ $\lambda$ and $\langle N, m(g) \rangle$ (for any $G$-local martingale $m(g)$) are $G$-predictable,
- $E')$ $H$ is a $G_T$-measurable bounded random variable,
- $F')$ $\int_0^T \lambda_t^2 d\langle N \rangle_t \leq C$, $P - \text{a.s.}$

Similarly to previously, we can prove that under conditions $A'), B'), C')$

- $\hat{S}_t = E(S_t|G_t) = S_0 + \int_0^t \hat{\lambda}_u d\langle N \rangle_u + \hat{N}_t,$
- $\hat{N}_t = \int_0^t (\frac{d\langle N, m(g) \rangle_u}{dm(g)_u}) dm_u(g) + L_t(g)$, for any $G$ local martingale $m(g)$.
- $\langle N, m(g) \rangle^G = \langle \hat{N}, m(g) \rangle$ for any $G$ local martingale $m(g)$, where $A^G$ denotes the dual $G$-predictable projection of $A$. 
Now, we give necessary and sufficient conditions for $G$ to be sufficient.

**Proposition** Let conditions $A', B', C', F'$ be satisfied. Then the two value processes $V_t(A)$ and $V_t(G)$ coincide if and only if $H$ is $G_T$-measurable and bounded and the process $V_t(A)$ satisfies the BSDE

$$Y_t = Y_0 + \frac{1}{2} \int_0^t \left( \psi_u k_u^2 + \lambda_u Y_u \right) \frac{d\langle N\rangle_u}{Y_u} + \int_0^t \psi_u d\hat{N}_u + \tilde{N}_t, \quad Y_T = e^{\alpha H}, \quad (12)$$

where $\tilde{N}$ is a $G$-local martingale orthogonal to $\hat{N}$ and $k_t^2 = d\langle \hat{N} \rangle_t / d\langle N \rangle_t$.

- The “necessity part” relies essentially on the decomposition of $\hat{N}$ which enables to prove that, if $V_t(A) = V_t(G)$, the BSDE satisfied by $V_t(A)$ can be written as Eq. (12).

**Remark** Note that we do not require condition $D'$, which is needed to find “the equivalent problem” in the partial information case.
• If $\mathcal{F}^S \subseteq \mathcal{G}$, we can reduce the necessary and sufficient conditions as follows.

**Corollary** If $\mathcal{F}^S \subseteq \mathcal{G}$ and $C'$ and $F'$ be satisfied. Then

a) $V_t(\mathcal{A}) = V_t(\mathcal{G})$ if and only if $H$ is $\mathcal{G}_T$-measurable and bounded and $\lambda$ is $\mathcal{G}$-predictable.

b) If in addition condition $B'$ is satisfied and $H$ is bounded and $\mathcal{G}_T$-measurable, then $V_0(\mathcal{A}) = V_0(\mathcal{G})$ if and only if $\lambda$ is $\mathcal{G}$-predictable.

• Relying on the previous results, we have

**Theorem** Let conditions $A'$–$F'$ be satisfied. Then $V_t(\mathcal{A}) = V_t(\mathcal{G})$ and the filtration $\mathcal{G}$ is sufficient.

**Remark** Note that under conditions $A)$–$F$ the filtration $\mathcal{G}$ is sufficient for the auxiliary filtration $\mathcal{F} = \mathcal{F}^S_t \vee \mathcal{G}$, i.e. for the optimization problem $\mathcal{O}\Pi = \{ (\Omega, \mathcal{F}, \mathcal{F}, P), S, H \}$.  


Thank you.
References


• V. Henderson and D.G. Hobson, Real options with constantrelative risk aversion, Journal of Economic Dynamics and Control 29, N.7, 2005, 1237–1266


• M. Musiela and T. Zariphopoulou, An example of indifference prices under exponential preferences, Finance Stochast. 8, 2004, 229–239.

